4.1 Vector Spaces $\mathbb{R}^2$ and $\mathbb{R}^3$

- Watch video "linear combinations, span, ..." (linked)
- Watch video "linear transformations ..." (linked)

(both by Youtube channel 3Blue1Brown)

Set notation $\left\{ \left(\begin{matrix}\text{generic}\\ \text{element}\end{matrix}\right) : \left(\begin{matrix}\text{specific}\\ \text{conditions}\end{matrix}\right) \right\}$

often specify "pool/universe"

"such that"

narrow down "pool"

"in"

$$\{ x \in \mathbb{R} : x > 0 \} = (0, \infty)$$

$$\{ (x,y) \in \mathbb{R}^2 : x>0, y<0 \} = \text{"}4^{th} \text{ quadrant"}$$

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \in \mathbb{R}, y=0 \right\} = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\} \text{"x axis"}$$

- Sets discard duplicate elements

Linear combination : a weighted sum of vectors.

Ex: $2\underline{u} - 3\underline{v}$, $\frac{1}{3}\underline{u} + 5\underline{v} + \pi \cdot \underline{w}$, ...

Span : set of all linear combinations :

$$\text{span}\{\underline{u}, \underline{v}\} = \{ a\underline{u} + b\underline{v} : a, b \in \mathbb{R} \}$$

- $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \left\{ a\begin{bmatrix} 1 \\ 1 \end{bmatrix} : a \in \mathbb{R} \right\} = (\text{line } y = x)$
- $\text{span}\{\hat{\imath}, \hat{\jmath}\} = \mathbb{R}^2$ (all points/vectors covered)
- $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \left\{ a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\} = \mathbb{R}^2$

- $\text{span}\left\{\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}0\\1\\-2\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\0\\3\end{bmatrix} + b\begin{bmatrix}0\\1\\-2\end{bmatrix} : a,b \in \mathbb{R}\right\}$

$$= \left\{\begin{bmatrix}a\\b\\3a-2b\end{bmatrix} : a,b \in \mathbb{R}\right\}$$

$$= \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : x,y \in \mathbb{R},\ \underline{z = 3x-2y}\right\}$$

$3x-2y-z=0$ (a plane)

Def: if $\{\underline{u}, \underline{v}\}$ "span" a set S, they are a "<u>spanning set</u>" for S.

- $\{\hat{\imath}, \hat{\jmath}\}$ is a spanning set for $\mathbb{R}^2$, but <u>not</u> for $\mathbb{R}^3$
- $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ is a " " " " for $\mathbb{R}^2$

Def: $\underline{u}$ and $\underline{v}$ are <u>linearly dependent</u> if $\underline{u} = c\underline{v}$ or $\underline{v} = c\underline{u}$ (or both).

If not, they are <u>lin.</u> <u>independent</u>

"$\{\underline{u}, \underline{v}\}$ is a <u>lin.</u> <u>ind./dep.</u> set"

Def: 3 or more vectors are <u>lin.</u> <u>dep.</u> if some one can be made as a <u>lin.</u> <u>comb.</u> of the others*. If not, they are <u>lin. indep.</u>

(* alt: if one is in the span of the others)

Ex $\underline{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$, $\underline{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Try to make $a\underline{u} + b\underline{v} = \underline{w}$:

Want $a\begin{bmatrix} -1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$(A\underline{x} = \underline{b}) \longrightarrow \begin{bmatrix} -1 & 7 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Solve using (RREF) or (EF) ... $\underline{x} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$

so $\boxed{-\frac{1}{2}\underline{u} + \frac{1}{2}\underline{v} = \underline{w},}$. $\left(\underline{w} \in \text{span}\{\underline{u}, \underline{v}\}\right)$

Also, $\underline{v} = \underline{u} + 2\underline{w}$, or $\underline{u} = \underline{v} - 2\underline{w}$.

With more vectors, "which is the lin. comb.?"

harder to guess, and ambiguous.

So, easier to just say lin. comb.

$$\boxed{\underline{u} - \underline{v} + 2\underline{w} = \underline{0}}$$

Is $\underbrace{\begin{bmatrix} -2 \\ 5 \\ -13 \end{bmatrix}}_{\underline{w}} \in \text{span}\left\{\underbrace{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}_{\underline{u}}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}}_{\underline{v}}\right\}$?

$\iff$ solution to $a\underline{u} + b\underline{v} = \begin{bmatrix} -2 \\ 5 \\ -13 \end{bmatrix}$

$$\begin{array}{c} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 3 & -2 & -13 \end{array}\right] \\ \begin{array}{cc} a & b \end{array} \end{array} \longrightarrow \begin{array}{c} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{array}\right] \\ \begin{array}{cc} a & b \end{array} \end{array} \quad 0 = 3 \ldots$$

no solution, so $\begin{bmatrix} -2 \\ 5 \\ -13 \end{bmatrix} \underset{\text{(not in)}}{\notin} \text{Span}\{\underline{u}, \underline{v}\}$

Def: (Indepence): Given $\underline{u}, \underline{v}, \underline{w}$, try to make

lin. comb. $\quad a\underline{u} + b\underline{v} + c\underline{w} = \underline{0}$.

Always possible with "trivial solution"

$$(a,b,c) = (0,0,0).$$

1) If that is the only lin. comb. making $\underline{0}$,
   then $\underline{u}, \underline{v}, \underline{w}$ are lin. indep.

2) If there are other combinations making $\underline{0}$,
   they are lin. dep.